3965. Proposed by Ovidiu Furdui and Alina Sîntămărian.

Determine the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\left(\ln\frac{1}{2}+1-\frac{1}{2}+\frac{1}{3}-\ldots+\frac{(-1)^n}{n}\right)x^n$$

and its value at x for each x in this interval.

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Firstly we will prove that power series convergence in the interval $(-1,1)$.

Let $S_n := \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}-\ln 2$. Since $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=\ln 2$ then $S_n=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}-\ln 2=$

$$-\sum_{k=1}^{\infty}\frac{(-1)^{n+k-1}}{n+k}=(-1)^{n-1}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k}=(-1)^{n-1}\sum_{k=1}^{\infty}\int_0^1(-1)^{k-1}t^{n+k-1}dt=$$

$$(-1)^{n-1}\int_0^1\left(\sum_{k=1}^{\infty}(-1)^{k-1}t^{n+k-1}\right)dt=(-1)^{n-1}\int_0^1 t^n\left(\sum_{k=1}^{\infty}(-1)^{k-1}t^{k-1}\right)dt=(-1)^{n-1}\int_0^1\frac{t^n}{1+t}dt.$$

Thus, $|S_n|=\int_0^1\frac{t^n}{1+t}dt$. Since $\frac{t^n}{2}\le\frac{t^n}{1+t}\le t^n$ for $t\in[0,1]$ then

$$\frac{1}{2(n+1)}\le\int_0^1\frac{t^n}{1+t}dt\le\frac{1}{n+1}$$

and, therefore, $\lim_{n\to\infty}\sqrt[n]{\frac{1}{2(n+1)}}\le\lim_{n\to\infty}\sqrt[n]{|S_n|}\le\lim_{n\to\infty}\sqrt[n]{\frac{1}{n+1}}\Rightarrow\lim_{n\to\infty}\sqrt[n]{|S_n|}=1$

(because $\lim_{n\to\infty}\sqrt[n]{\frac{1}{2(n+1)}}=\lim_{n\to\infty}\sqrt[n]{\frac{1}{n+1}}=1$).

Then by Cauchy's radical test we obtain that interval of convergence of the power series $\sum_{n=1}^{\infty}S_nx^n$ is $(-1,1)$.

Let $x\in(-1,1)$. Then $\sum_{n=1}^{\infty}\left(\ln\frac{1}{2}+1-\frac{1}{2}+\frac{1}{3}-\ldots+\frac{(-1)^{n-1}}{n}\right)x^n=\sum_{n=1}^{\infty}S_nx^n=$

$$\frac{1}{1-x}\sum_{n=1}^{\infty}S_n(x^n-x^{n+1})=\frac{1}{1-x}\left(\sum_{n=1}^{\infty}S_nx^n-\sum_{n=1}^{\infty}S_nx^{n+1}\right)=\frac{1}{1-x}\left(\sum_{n=1}^{\infty}S_nx^n-\sum_{n=2}^{\infty}S_{n-1}x^n\right)=$$

$$\frac{1}{1-x}\left(S_1x+\sum_{n=2}^{\infty}(S_n-S_{n-1})x^n\right)=\frac{1}{1-x}\left(S_1x+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n}x^n\right)=$$

$$\frac{1}{1-x}\left(x-x\ln 2+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n}x^n\right)=\frac{1}{1-x}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n-x\ln 2\right)=\frac{\ln(1+x)-x\ln 2}{1-x}.$$

Remark.

Since $S_n=(-1)^{n-1}\int_0^1\frac{t^n}{1+t}dt$ then for $x=1$ we obtain series $\sum_{n=1}^{\infty}S_n=\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where $a_n:=\int_0^1\frac{t^n}{1+t}dt$. Since $\lim_{n\to\infty}a_n=0$ then by Leibniz Test series $\sum_{n=1}^{\infty}S_n$ conditionally convergent.

If $x=-1$ then we obtain series $\sum_{n=1}^{\infty}S_n(-1)^n=-\sum_{n=1}^{\infty}a_n$ which diverges because $a_n\ge\frac{1}{2(n+1)}$.